

# WILF CLASSIFICATION OF THREE AND FOUR LETTER SIGNED PATTERNS

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ABSTRACT. We give some new Wilf equivalences for signed patterns which allow the complete classification of signed patterns of lengths three and four. The problem is considered for pattern avoidance by general as well as involutive signed permutations.

## 1. INTRODUCTION

Let  $S_n$  and  $B_n$  denote the symmetric group and hyperoctahedral group on  $[n] = \{1, 2, \dots, n\}$ , respectively. We regard the elements of  $B_n$  as signed permutations written as a word  $\pi = \pi_1\pi_2 \dots \pi_n$  in which each of the integers  $1, 2, \dots, n$  appears, possibly signed which is represented by a bar. Throughout this paper, we use the terms “barred/unbarred” and “negative/positive” synonymously. Furthermore let  $I_n$  and  $SI_n$  be the set of involutions in  $S_n$  and  $B_n$ , respectively.

The *barring operation* maps  $\pi \in B_n$  to the signed permutation  $\bar{\pi}$  which is obtained from  $\pi$  by changing the sign of all elements. Clearly, we have  $\bar{\bar{\pi}} = \pi$ . The absolute value notation means  $|\pi_i| = \pi_i$  if  $\pi_i$  is not barred and  $|\pi_i| = \bar{\pi}_i$  otherwise. In particular, we write  $|\pi|$  for the permutation  $|\pi_1||\pi_2| \dots |\pi_n|$ .

First we recall the concept of pattern-avoidance for signed permutations. A signed permutation  $\pi \in B_n$  is said to *contain the pattern*  $\tau \in B_k$  if there exists a sequence  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  such that  $|\pi_{i_a}| < |\pi_{i_b}|$  if and only if  $|\tau_a| < |\tau_b|$  and  $\pi_{i_a} > 0$  if and only if  $\tau_a > 0$  for all  $1 \leq a, b \leq k$ . Otherwise,  $\pi$  is called a  $\tau$ -*avoiding* permutation. By  $M(\tau)$  we denote the set of all elements of  $M$  which avoid the pattern  $\tau$ , and by  $|M(\tau)|$  its cardinality.

Two signed patterns  $\sigma$  and  $\tau$  are called Wilf equivalent, in symbols  $\sigma \sim \tau$ , if they are avoided by the same number of signed  $n$ -permutations for each  $n \geq 1$ . When we consider avoidance only in the set of signed involutions then we use  $\stackrel{I}{\sim}$  to denote the equivalence.

Obviously, if  $\sigma, \tau \in S_k$  are Wilf equivalent patterns in  $S_k$ , i. e.  $|S_n(\sigma)| = |S_n(\tau)|$  for all  $n \geq 1$ , then they are also equally restrictive for  $B_n$ .

The classification of patterns up to Wilf equivalence is a basic problem in the theory of forbidden

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subsequences. Whereas many equivalences can be simply put down to symmetries within the symmetric and hyperoctahedral groups, detecting and proving *all* Wilf equivalences in  $S_k$  or  $B_k$  is a difficult task which requires a more subtle approach. Over a period of 20 years, the classification of permutation patterns up to length 7 has been successfully completed. The bijection between  $S_n(123)$  and  $S_n(132)$  given by Simion and Schmidt [8] initiated this; it showed the whole of  $S_3$  to be one Wilf class. The classification of  $S_4$  turned out to be considerably more complicated and was done by West and Stankova in the series of papers [12], [9], [10]. Using a stronger equivalence relation, Babson and West [1] showed that the patterns  $(k-1)k\tau$  and  $k(k-1)\tau$  for any  $\tau \in S_{k-2}$  and  $(k-2)(k-1)k\tau$  and  $k(k-1)(k-2)\tau$  for any  $\tau \in S_{k-3}$  are equally restrictive which provides the classification of  $S_5$ . However, the generalization  $(k-l+1)(k-l+2)\dots k\tau \sim k(k-1)\dots(k-l+1)\tau$  for any  $\tau \in S_{k-l}$  given by Backelin, West, and Xin [2] was not sufficient to classify  $S_6$  completely. The last open case was settled by the equivalence of  $(k-1)(k-2)k\tau$  and  $(k-2)k(k-1)\tau$ , shown in Stankova and West [11]. With that it was even possible to complete the classification of patterns of length 7 which is the current limit.

Analogous investigations for pattern avoidance by involutive permutations were conducted in Guibert [5], Jaggard [6] and Bousquet-Mélou and Steingrímsson [3].

In this paper the classification problem is completed for signed patterns of length at most 4. Simion [7] proved that all elements of  $B_2$  are Wilf equivalent to each other if pattern avoidance by all signed permutations is considered. For signed involutions,  $B_2$  falls into two Wilf classes, see [4]. We determine the Wilf classes for three and four letter signed patterns both regarding avoidance by general and involutive signed permutations.

## 2. WILF EQUIVALENCES FOR SIGNED PATTERNS

Before enumerating signed pattern-avoiding permutations, we shall study the equivalences between signed patterns. This will decrease considerably the number of patterns that need individual attention. There are many trivial equivalences based on symmetries. For a signed permutation  $\pi = \pi_1\pi_2\dots\pi_n$ , we define its *reverse* as the permutation  $\pi_n\pi_{n-1}\dots\pi_1$  and its *complement* as the permutation whose  $i$ th element is  $n+1-\pi_i$  if  $\pi_i$  is positive and  $-(n+1)-\pi_i$  otherwise. It is obvious that two signed patterns are Wilf equivalent if one of them can be transformed into the other by a sequence of reverse, complement or barring operations. (Note that the symmetry group is smaller if we consider the problem in the set of signed involutions. The group is then generated by the barring operation and the composition of reverse and complement operations.) Furthermore, every pattern is obviously Wilf equivalent to its inverse since  $\tau$  is avoided by  $\pi$  if and only if  $\tau^{-1}$  is avoided by  $\pi^{-1}$ .

In this section we will give some nontrivial equivalences which are the key to the complete classification of short signed patterns. Our bijective proofs use essentially the properties of

right-to-left maxima of permutations. The idea to consider these special elements is based on the classical bijection given by [8] for patterns of length three.

An element  $\pi_i$  of a word  $\pi$  is called a *right-to-left maximum* if it is greater than all elements that follow it, i.e.  $\pi_i > \pi_j$  for all  $j > i$ . We define successively the  $r$ -right-to-left maxima for a signed permutation  $\pi \in B_n$ . Let  $\pi^{(1)}$  be the subword consisting of all unbarred elements of  $\pi$ . For  $r \geq 1$ , the right-to-left maxima of  $\pi^{(r)}$  are called  *$r$ -right-to-left maxima* of  $\pi$  where  $\pi^{(r+1)}$  is the subword obtained from  $\pi^{(r)}$  by removing all  $r$ -right-to-left maxima.

For example, the signed permutation  $\pi = 2\bar{5}6310\bar{8}417\bar{9} \in B_{10}$  has the 1-right-to-left maxima 10 and 7; the 2-right-to-left maxima 6, 4, and 1; the 3-right-to-left maximum 3; and the 4-right-to-left maximum 2.

Note that the  $r$ -right-to-left maxima of  $\pi$  form a decreasing subsequence for each  $r$ . Furthermore, each  $r$ -right-to-left maximum  $a$  is the initial term of an increasing subsequence of unbarred elements in  $\pi$  of length  $r$ , and there is no increasing subsequence of unbarred elements of length  $r + 1$  which starts with  $a$ .

Our first result states that barring the prefix of an increasing pattern yields an equivalence.

**Theorem 2.1.** *For any pattern  $\tau \in B_l$  and  $k \geq l$ , we have*

$$\tau(l+1)(l+2)\dots k \sim \bar{\tau}(l+1)(l+2)\dots k.$$

*Moreover, the patterns are also equivalent if we consider their avoidance by signed involutions.*

*Proof.* We construct a bijection  $\Phi_{k-l}$  from the set of all  $\tau(l+1)(l+2)\dots k$ -avoiding permutations in  $B_n$  to the set of all  $\bar{\tau}(l+1)(l+2)\dots k$ -avoiding permutations in  $B_n$  which preserves all  $(k-l)$ -right-to-left maxima. In addition,  $\Phi_{k-l}$  maps involutions to involutions again.

Given a signed permutation  $\pi \in B_n(\tau(l+1)(l+2)\dots k)$ , we define  $\sigma = \Phi_{k-l}(\pi)$  to be the permutation obtained from  $\pi$  by barring all elements  $\pi_i$  having a  $(k-l)$ -right-to-left maximum to their right which is greater than  $|\pi_i|$ . (This applies for an unbarred element  $\pi_i$  if and only if  $\pi_i$  is an  $r$ -right-to-left maximum with  $r > k-l$ .)

Why does the map leave all  $(k-l)$ -right-to-left maxima fixed? Let  $a$  be any  $(k-l)$ -right-to-left maximum of  $\pi$ . Then all elements following  $a$  whose absolute value is greater than  $a$  are left unchanged under  $\Phi_{k-l}$  because the  $(k-l)$ -right-to-left maxima decrease. Thus, by construction,  $a$  is also a  $(k-l)$ -right-to-left maximum of  $\sigma$ . (By similar reasoning, it can be shown that all the  $r$ -right-to-left maxima for  $r \leq k-l$  are preserved.)

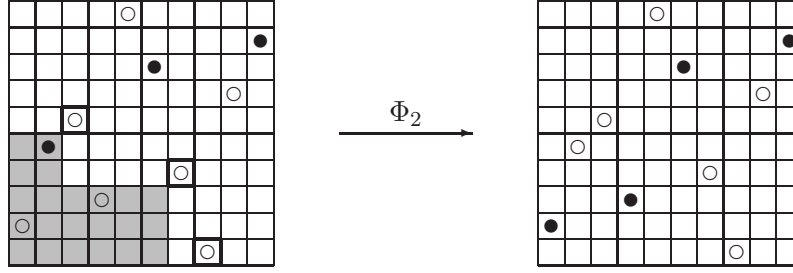
Since we have  $|\pi_i| = |\sigma_i|$  for all  $i$  and all the  $(k-l)$ -right-to-left maxima are preserved,  $\Phi_{k-l}$  is an involution and hence bijective.

Why does  $\sigma$  avoid the pattern  $\bar{\tau}(l+1)(l+2)\dots k$ ? We consider the  $(k-l)$ -right-to-left maximum  $a$  of  $\pi$  again. Note that an increasing unbarred subsequence of length  $k-l$  starts in  $a$ . Consequently, there is no subsequence  $\pi_{i_1}, \dots, \pi_{i_l}$  of elements preceding  $a$  with  $|\pi_{i_j}| < a$  for all  $j$  which is order-isomorphic to  $\tau$ . By applying  $\Phi_{k-l}$ , the element  $a$  is unchanged while all elements with

an absolute value smaller than  $a$  to the left of  $a$  are barred. Therefore there is no subsequence  $\sigma_{i_1}, \dots, \sigma_{i_l}$  formed of these elements which is order-isomorphic to  $\bar{\tau}$ .

Why is  $\sigma$  an involution whenever  $\pi$  is it? By construction,  $\Phi_{k-l}$  has an effect on the sign but not on the position of the elements. Let  $\pi_i = j$ . (Clearly,  $\pi_i$  and  $\pi_j$  have the same sign since  $\pi$  is an involution.) If there is a  $(k-l)$ -right-to-left maximum  $a$  to the right of  $\pi_i$  with  $a > |\pi_i|$  then the element  $b = \pi_a^{-1}$  is a  $(k-l)$ -right-to-left maximum to the right of  $\pi_j$  satisfying  $b > |\pi_j|$ . Thus  $\Phi_{k-l}$  changes the sign of  $\pi_i$  if and only if it does so for  $\pi_j$ .  $\square$

**Example 2.2.** Let  $\pi = 2\bar{5}6310\bar{8}417\bar{9} \in B_{10}$  again and  $k-l = 2$ . The following figure shows the effect of  $\Phi_2$ . We use the usual array representation of permutations with an additional colouring to make a distinction regarding the element sign. Barred elements are represented by black points while unbarred elements are represented by white points. The 2-right-to-left maxima are bordered. All the elements (points) which have to change their sign (colour) are contained in the grey region (the union of the south-west regions of all 2-right-to-left maxima).



We obtain  $\sigma = \bar{2}56\bar{3}10\bar{8}417\bar{9} \in B_{10}$  which has the same 2-right-to-left maxima as  $\pi$ .

**Corollary 2.3.** Let  $\tau \in B_k$  with  $|\tau| = 12 \dots k$ . Then  $\tau \sim 12 \dots k$  and even  $\tau \stackrel{I}{\sim} 12 \dots k$ .

*Proof.* Let  $l$  be the maximal integer with  $\tau_l = \bar{l}$ . We may assume that  $l < k$ ; otherwise we consider  $\bar{\tau}$  which is trivially Wilf equivalent to  $\tau$ . By Theorem 2.1, the patterns  $\tau_1 \dots \tau_l(l+1)(l+2) \dots k$  and  $\bar{\tau}_1 \dots \bar{\tau}_l(l+1)(l+2) \dots k$  are Wilf equivalent (also when considering signed involutions only). Induction on  $l$  yields the assertion.  $\square$

Our next result shows that barring the prefix of a pattern also yields an equivalent pattern if the monotone part is a decreasing sequence.

**Theorem 2.4.** For any pattern  $\tau \in B_l$  and  $k \geq l$ , we have

$$\tau k(k-1) \dots (l+1) \sim \bar{\tau} k(k-1) \dots (l+1).$$

*This relation remains true when we consider the pattern-avoiding signed involutions only.*

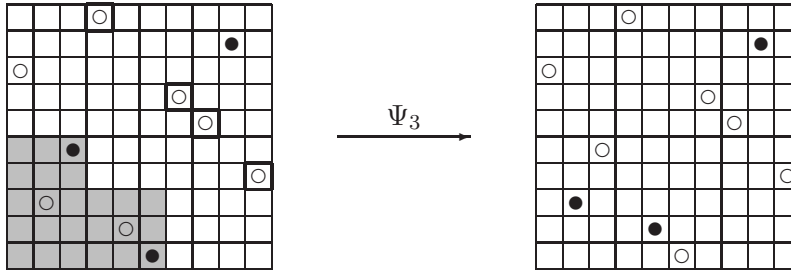
*Proof.* This proof closely follows that of Theorem 2.1, however the way in which we identify those elements whose sign will change differs slightly. Let  $\pi \in B_n(\tau k(k-1) \dots (l+1))$  and define  $\sigma = \Psi_{k-l}(\pi)$  to be the permutation obtained from  $\pi$  by changing the sign of all elements  $\pi_i$  having (at least)  $k-l$  1-right-to-left maxima to their right which are greater than  $|\pi_i|$ . (In

the graph of  $\pi$ , these are just the points  $(i, \pi_i)$  for which there is a decreasing sequence of  $k - l$  white points in the region north-east of  $(i, \pi_i)$ .

By definition, there are no unbarred elements in the north-east of any 1-right-to-left maximum; thus these maxima are fixed under  $\Psi_{k-l}$  (and hence also 1-right-to-left maxima of  $\sigma$ ). Consequently,  $\Psi_{k-l}(\sigma) = \pi$  and hence  $\Psi_{k-l}$  is a bijection between  $B_n(\tau k(k-1) \dots (l+1))$  and  $B_n(\bar{\tau} k(k-1) \dots (l+1))$ .

Obviously,  $\sigma$  avoids  $\bar{\tau} k(k-1) \dots (l+1)$ . If we suppose otherwise, then there exist a sequence  $a_1 a_2 \dots a_k$  in  $\sigma$  that is order-isomorphic to  $\bar{\tau} k(k-1) \dots (l+1)$ . (We may assume that  $a_{l+1}, \dots, a_k$  are 1-right-to-left maxima of  $\sigma$ .) Since  $a_{l+1} > a_{l+2} > \dots > a_k > |a_i|$  for all  $i \in [l]$ , we must have  $\bar{a}_1 \dots \bar{a}_l a_{l+1} \dots a_k$  in  $\pi$ , and this is an occurrence of  $\tau k(k-1) \dots (l+1)$ , which is false. The proof remains true when  $\sim$  is replaced by  $\overset{I}{\sim}$  since  $\Psi_{k-l}$  maps signed involutions to signed involutions.  $\square$

**Example 2.5.** Consider the signed permutation  $\pi = 8\ 3\ \bar{5}\ 10\ 2\ \bar{1}\ 7\ 6\ \bar{9}\ 4 \in B_{10}$  which avoids the pattern  $\bar{1}\ 2\ \bar{5}\ 4\ 3$ . Its 1-right-to-left maxima are 10, 7, 6, and 4.



The shaded region highlights the positions which are subject to sign-change. We obtain  $\Psi_3(\pi) = 8\ 3\ \bar{5}\ 10\ \bar{2}\ \bar{1}\ 7\ 6\ \bar{9}\ 4 \in B_{10}(1\ \bar{2}\ \bar{5}\ 4\ 3)$ .

The third equivalence which we will use for the Wilf classification in the next sections is an immediate adjustment of a result for pattern of the symmetric group given by Backelin, West, and Xin [2]. They proved that the monotone suffix of any pattern can be reversed under Wilf equivalence. Bousquet-Mélou and Steingrímsson [3] showed that this equivalence is preserved when considering avoidance by involutions.

**Theorem 2.6.** *For any pattern  $\tau \in B_l$  and  $k \geq l$ , we have*

$$\tau(l+1)(l+2) \dots k \sim \tau k(k-1) \dots (l+1)$$

where  $\sim$  can be replaced by  $\overset{I}{\sim}$ .

### 3. CLASSIFICATION OF SIGNED PATTERNS OF LENGTH THREE

There are 48 signed patterns of length three but by taking all symmetries in consideration, we can restrict our attention to the following six: 123,  $\bar{1}23$ ,  $\bar{1}\bar{2}3$ , 132,  $\bar{1}32$ ,  $\bar{1}\bar{3}2$ . It is well known that all patterns of  $S_3$  are avoided by the same number of permutations in  $S_n$ . Hence these

patterns are also Wilf equivalent in  $B_3$ . This result and Theorem 2.1 and 2.4 reduce the number of patterns which shall be considered up to two. (We have  $123 \sim \bar{1}23 \sim \bar{1}\bar{2}3 \sim 132 \sim \bar{1}32$ .) A look at the initial terms of the sequences  $(|B_n(\tau)|)_{n \geq 0}$  yields the Wilf classification of  $B_3$ .

123	1 2 8 47 358 3312 35784 440001
$\bar{1}32$	1 2 8 47 358 3311 35738 438561

TABLE 1. Wilf classes of  $B_3$  (avoidance by  $B_n$ )

Now we compare the cardinalities of the sets  $SI_n(\tau)$  for  $\tau \in B_3$ . By symmetry and inversion, we obtain twelve patterns, namely 123,  $\bar{1}23$ ,  $\bar{1}\bar{2}3$ , 132,  $\bar{1}32$ , 321 (which are all equivalent by Theorem 2.1, 2.4, and 2.6),  $\bar{1}\bar{3}2$ , 231,  $\bar{2}31$ ,  $23\bar{1}$ ,  $\bar{3}21$ ,  $3\bar{2}1$ .

**Proposition 3.1.** *We have  $321 \stackrel{I}{\sim} 3\bar{2}1$ .*

*Proof.* Let  $\pi \in SI_n(321)$  and  $\varphi$  be the map that changes the sign of all (necessarily barred) elements  $\pi_i$  for which there are unbarred elements  $\pi_j$  and  $\pi_k$  with  $j < i < k$  and  $\pi_j > |\pi_i| > \pi_k$ . (Therefore,  $\pi_i$  is the middle element of an occurrence of  $3\bar{2}1$  in  $\pi$ .) Obviously,  $\varphi(\pi)$  is a  $3\bar{2}1$ -avoiding involution. Note that  $jik$  is an occurrence of  $3\bar{2}1$  whenever  $\pi_j\pi_i\pi_k$  is it. It is clear that  $\varphi^2(\pi) = \pi$  and  $\varphi$  is therefore a bijection.  $\square$

Using the same bijection, we can generalize the result as follows.

**Corollary 3.2.** *For any integers  $k, s \geq 1$  and any permutation  $\tau$  on  $\{s+1, s+2, \dots, s+k\}$  we have*

$$(2s+k) \dots (s+k+1) \tau s \dots 1 \stackrel{I}{\sim} (2s+k) \dots (s+k+1) \bar{\tau} s \dots 1.$$

*Proof.* The case  $s = 1$  is proved by the bijection  $\varphi$  established in the previous proof. (Note that the elements  $\pi_i$  need not be barred now.) Clearly,  $\pi \in SI_n$  avoids  $(k+2)\tau 1$  if and only if  $\varphi(\pi)$  avoids  $(k+2)\bar{\tau} 1$ . Induction on  $s$  yields the assertion.  $\square$

**Remark 3.3.** The statement is also correct if we consider the relation  $\sim$  instead of  $\stackrel{I}{\sim}$ . (Of course, the map  $\varphi$  can be applied to  $B_n$  as well.) However, for general signed permutations this equivalence is already covered by Theorem 2.1 since  $(k+2)\tau_2 \dots \tau_{k+1} 1$  and  $1\tau_{k+1} \dots \tau_2(k+2)$  are related by symmetry. For instance, we have

$$53241 \stackrel{s}{\sim} 14235 \stackrel{*}{\sim} \bar{1}\bar{4}\bar{2}\bar{3}5 \stackrel{s}{\sim} \bar{1}4235 \stackrel{*}{\sim} 1\bar{4}\bar{2}\bar{3}5 \stackrel{s}{\sim} 5\bar{3}\bar{2}\bar{4}1$$

where  $\stackrel{s}{\sim}$  stands for a symmetry relation and  $\stackrel{*}{\sim}$  means equivalence by Theorem 2.1.

The computation of the initial terms of the sequences  $(|SI_n(\tau)|)_{n \geq 0}$  for the remaining six patterns shows that there are no further equivalences.

123	1 2 6 19 68 256 1032 4341 19154 87604 415868
132	1 2 6 20 74 288 1178 4978 21738 97420 448172
231	1 2 6 20 74 292 1220 5336 24316 114872 560840
$\bar{2}31$	1 2 6 20 74 291 1207 5215 23362 107960 513236
$2\bar{3}1$	1 2 6 20 75 299 1259 5501 24813 114729 542074
$\bar{3}21$	1 2 6 20 75 298 1250 5430 24347 111821 524921

TABLE 2. Wilf classes of  $B_3$  (avoidance by  $SI_n$ )

#### 4. CLASSIFICATION OF SIGNED PATTERNS OF LENGTH FOUR

Extending the pattern length to four increases the complexity considerably. The 384 permutations in  $B_4$  can be partitioned into 40 symmetry classes represented by:

$$\begin{aligned}
&1234, \bar{1}234, \bar{1}2\bar{3}4, \bar{1}2\bar{3}\bar{4}, \bar{1}2\bar{3}4, 1324, \bar{1}324, \bar{1}3\bar{2}4, \bar{1}3\bar{2}\bar{4}, 2134, \bar{2}134, 21\bar{3}4, \\
&\bar{2}\bar{1}34, 2\bar{1}\bar{3}4, \bar{2}\bar{1}\bar{3}4, 2143, \bar{2}143, \bar{2}\bar{1}43, \bar{2}\bar{1}\bar{4}3, 2\bar{1}\bar{4}3, 2314, \bar{2}314, \bar{2}3\bar{1}4, 23\bar{1}4, \bar{2}3\bar{1}4, \bar{2}3\bar{1}\bar{4} \\
&2\bar{3}\bar{1}4, \bar{2}3\bar{1}\bar{4}, 2413, \bar{2}413, \bar{2}\bar{4}13, 2\bar{4}\bar{1}3, 3214, \bar{3}214, 3\bar{2}14, \bar{3}2\bar{1}4, 3\bar{2}\bar{1}\bar{4}
\end{aligned}$$

The study of the patterns in  $S_4$  showed that they can be partitioned into only three Wilf classes, namely those of 1234, 1324, and 2314. These equivalences are, of course, also valid in  $B_4$ . Applying Theorems 2.1, 2.4, and 2.6 reduces the number of patterns which we still need to consider to the following:

$$\begin{aligned}
&1234 \sim 3214 \sim 2134 \sim 2143 \sim \bar{1}234 \sim \bar{1}2\bar{3}4 \sim \bar{1}2\bar{3}\bar{4} \sim \bar{1}2\bar{3}4 \sim \bar{2}\bar{1}34 \sim 21\bar{3}4 \sim \bar{2}\bar{1}\bar{3}4 \sim \bar{2}\bar{1}43 \sim \bar{3}2\bar{1}4, \\
&1324 \sim \bar{1}324 \sim \bar{1}3\bar{2}4 \quad (\text{since } \bar{1}324 \sim \bar{1}3\bar{2}\bar{4} \text{ by symmetry}), \\
&1\bar{3}24 \sim \bar{1}3\bar{2}4, \quad \bar{2}134 \sim 2\bar{1}\bar{3}4 \sim \bar{2}143, \quad 2314 \sim 2413 \sim \bar{2}3\bar{1}4, \quad \bar{2}314 \sim \bar{2}3\bar{1}4, \quad \bar{2}3\bar{1}4 \sim \bar{2}3\bar{1}\bar{4}, \\
&23\bar{1}4 \sim \bar{2}3\bar{1}4, \quad \bar{3}214 \sim 3\bar{2}\bar{1}4, \quad 3\bar{2}14 \sim \bar{3}2\bar{1}4, \quad \bar{2}\bar{1}\bar{4}3, \quad 2\bar{1}\bar{4}3, \quad \bar{2}413, \quad \bar{2}\bar{4}13, \quad 2\bar{4}\bar{1}3
\end{aligned}$$

**Proposition 4.1.** *We have  $\bar{2}\bar{1}\bar{4}3 \sim 2\bar{1}\bar{4}3$ .*

*Proof.* Let  $\pi \in B_n(\bar{2}\bar{1}\bar{4}3)$ . Define  $\psi$  to be the map that changes the sign of all elements  $\pi_i$  for which there is a sequence  $\pi_j\pi_k$  to the right of  $\pi_i$  with  $|\pi_i| < |\pi_k| < |\pi_j|$  and  $\pi_j < 0 < \pi_k$ . (That means,  $\pi_i$  is the first element of an occurrence of  $\bar{1}3\bar{2}$  or  $\bar{1}\bar{3}2$  in  $\pi$ .) Obviously,  $\psi(\pi)$  avoids  $2\bar{1}\bar{4}3$  since the first two elements of any occurrence of  $2\bar{1}\bar{4}3$  would be elements whose sign was changed by  $\psi$  and hence  $\pi$  would contain the pattern  $\bar{2}\bar{1}\bar{4}3$ . Clearly,  $\psi^2(\pi) = \pi$  and hence  $\psi$  is bijective.  $\square$

Table 3 lists the initial terms of the sequences  $(|B_n(\tau)|)_{n \geq 0}$  for the fourteen patterns  $\tau \in B_4$  which remain. For  $n = 7$ , they are all different; hence the classification is done.

Now we turn our attention to comparing the four letter patterns regarding their avoidance by signed involutions. Taking all symmetries for involutions into consideration, we obtain 78 classes.

1234	1 2 8 48 383 3798 44811 610354	1324	1 2 8 48 383 3798 44811 610355
1 $\bar{3}$ 24	1 2 8 48 383 3798 44809 610214	$\bar{2}$ 134	1 2 8 48 383 3798 44809 610280
$\bar{2}$ 1 $\bar{4}$ 3	1 2 8 48 383 3798 44810 610268	2314	1 2 8 48 383 3798 44810 610284
$\bar{2}$ 314	1 2 8 48 383 3798 44809 610212	$\bar{2}$ 314	1 2 8 48 383 3798 44809 610210
23 $\bar{1}$ 4	1 2 8 48 383 3798 44809 610277	$\bar{2}$ 413	1 2 8 48 383 3798 44809 610214
$\bar{2}$ 413	1 2 8 48 383 3798 44808 610144	$\bar{2}$ 4 $\bar{1}$ 3	1 2 8 48 383 3798 44808 610130
$\bar{3}$ 214	1 2 8 48 383 3798 44809 610279	$\bar{3}$ 214	1 2 8 48 383 3798 44809 610276

TABLE 3. Wilf classes of  $B_4$  (avoidance by  $B_n$ )

All the equivalences which we have obtained in the general case, apart from that one between 2314 and 2413 (given by Stankova [9]), are based on Theorem 2.1, 2.4, and 2.6. Therefore they are still valid. By computation, we will see that the equivalence of 2314 and 2413 gets lost for the involution case. So we may concentrate on the remaining cases:

1234, 1324, 1 $\bar{3}$ 24,  $\bar{2}$ 134,  $\bar{2}$ 1 $\bar{4}$ 3,  $\bar{2}$ 1 $\bar{4}$ 3, 2314,  $\bar{2}$ 314,  $\bar{2}$ 314, 23 $\bar{1}$ 4, 2413,  $\bar{2}$ 413,  $\bar{2}$ 4 $\bar{1}$ 3,  $\bar{2}$ 4 $\bar{1}$ 3,  $\bar{3}$ 214,  $\bar{3}$ 214;  
 $\bar{2}$ 413,  $\bar{2}$ 4 $\bar{1}$ 3, 3412,  $\bar{3}$ 412,  $\bar{3}$ 412,  $\bar{3}$ 4 $\bar{1}$ 2,  $\bar{3}$ 4 $\bar{1}$ 2, 4123,  $\bar{4}$ 123,  $\bar{4}$ 1 $\bar{2}$ 3,  $\bar{4}$ 1 $\bar{2}$ 3,  $\bar{4}$ 1 $\bar{2}$ 3,  $\bar{4}$ 1 $\bar{2}$ 3, 4213,  $\bar{4}$ 213,  $\bar{4}$ 2 $\bar{1}$ 3,  $\bar{4}$ 2 $\bar{1}$ 3,  $\bar{4}$ 2 $\bar{1}$ 3,  $\bar{4}$ 2 $\bar{1}$ 3,  $\bar{4}$ 2 $\bar{1}$ 3, 4231,  $\bar{4}$ 231,  $\bar{4}$ 231,  $\bar{4}$ 231,  $\bar{4}$ 2 $\bar{3}$ 1, 4312,  $\bar{4}$ 312,  $\bar{4}$ 312,  $\bar{4}$ 3 $\bar{1}$ 2,  $\bar{4}$ 3 $\bar{1}$ 2, 4321,  $\bar{4}$ 321,  $\bar{4}$ 321,  $\bar{4}$ 321,  $\bar{4}$ 3 $\bar{2}$ 1,  $\bar{4}$ 3 $\bar{2}$ 1.

The patterns in the latter part represent classes arising when we only consider symmetries of involutions and have to be studied individually now.

Using a result of Guibert [5], we have  $1234 \stackrel{I}{\sim} 3412 \stackrel{I}{\sim} 4321$ . (Hence the unsigned patterns  $\tau \in S_4$  contribute eight cases.) By Corollary 3.2, the relations  $4321 \stackrel{I}{\sim} 4\bar{3}\bar{2}1$  and  $4231 \stackrel{I}{\sim} 4\bar{2}\bar{3}1$  follow. Indeed, this completes the classification as Table 4 shows. We obtain here fifty Wilf classes. Note that the numbers  $|SI_n(\bar{2}1\bar{4}3)|$  and  $|SI_n(2\bar{1}\bar{4}3)|$  coincide for  $n \leq 9$ . This behaviour is quite exceptional; all the other patterns disclose their nonequivalence for shorter involutions.



1234	1 2 6 20 75 302 1299 5882 27899 137702 704716	1324	1 2 6 20 75 302 1299 5881 27889 137597 703878
1324	1 2 6 20 76 310 1354 6200 29644 146660 748752	2143	1 2 6 20 76 312 1378 6412 31246 157800 822452
2134	1 2 6 20 76 310 1356 6224 29880 148592 763532	2143	1 2 6 20 76 312 1378 6412 31246 157800 822448
2314	1 2 6 20 76 310 1358 6254 30202 151494 787398	2314	1 2 6 20 76 310 1358 6248 30117 150535 778460
2314	1 2 6 20 76 310 1357 6238 30022 149808 773051	2314	1 2 6 20 76 311 1368 6330 30676 154082 799383
2413	1 2 6 20 76 310 1360 6278 30444 153530 803578	2413	1 2 6 20 76 311 1370 6359 30994 156998 824015
2413	1 2 6 20 76 311 1370 6358 30971 156682 820465	2413	1 2 6 20 76 312 1381 6454 31678 161538 851968
2413	1 2 6 20 76 310 1359 6264 30290 152112 791459	2413	1 2 6 20 76 312 1380 6442 31566 160672 845866
3214	1 2 6 20 76 311 1367 6318 30560 153147 792385	3214	1 2 6 20 75 302 1300 5892 27993 138408 709859
3412	1 2 6 20 76 312 1378 6425 31428 159859 841636	3412	1 2 6 20 76 312 1382 6476 31924 163898 871838
3412	1 2 6 20 75 302 1298 5868 27750 136364 693620	3412	1 2 6 20 76 312 1380 6452 31704 162232 860414
4123	1 2 6 20 76 311 1368 6338 30797 155505 813216	4123	1 2 6 20 76 311 1370 6362 31015 157124 823967
4123	1 2 6 20 76 311 1368 6337 30775 155205 809915	4123	1 2 6 20 76 311 1369 6351 30924 156545 821054
4123	1 2 6 20 76 311 1369 6350 30903 156262 817929	4123	1 2 6 20 76 311 1368 6336 30758 154992 807670
4213	1 2 6 20 76 310 1358 6252 30176 151212 784880	4213	1 2 6 20 76 311 1370 6359 30977 156715 820350
4213	1 2 6 20 76 311 1368 6335 30754 155017 808670	4213	1 2 6 20 76 312 1380 6444 31592 160973 848763
4213	1 2 6 20 76 312 1380 6443 31573 160722 845999	4213	1 2 6 20 76 311 1368 6333 30719 154585 804274
4213	1 2 6 20 76 311 1369 6347 30866 155873 814600	4213	1 2 6 20 76 310 1358 6250 30140 150763 780284
4231	1 2 6 20 75 302 1299 5883 27911 137833 705870	4231	1 2 6 20 76 312 1379 6435 31510 160378 844431
4231	1 2 6 20 75 302 1299 5882 27897 137674 704384	4231	1 2 6 20 76 312 1378 6422 31380 159278 835774
4312	1 2 6 20 76 311 1368 6341 30840 155986 817676	4312	1 2 6 20 76 312 1380 6449 31661 161742 855816
4312	1 2 6 20 76 311 1369 6352 30936 156664 821993	4312	1 2 6 20 76 311 1369 6351 30918 156433 819537
4312	1 2 6 20 76 312 1382 6472 31872 163336 866840	4312	1 2 6 20 76 311 1368 6339 30804 155530 812915
4321	1 2 6 20 76 311 1369 6347 30859 155752 813020	4321	1 2 6 20 76 310 1358 6254 30200 151468 787094
4321	1 2 6 20 76 312 1382 6468 31820 162774 861850	4321	1 2 6 20 76 310 1360 6274 30374 152658 794576

TABLE 4. Wilf classes of  $B_4$  (avoidance by  $SI_n$ )

## 5. FINAL REMARKS

The application of all our results and the known Wilf classification of  $S_5$  to suited representatives of the symmetry classes of signed patterns of length 5 yields 137 patterns whose (non)equivalence have to be proved. By computer checks up to  $n = 8$ , it can be shown that these patterns form at least 58 different Wilf classes. It may be the case that the actual number of classes is much greater. By way of comparison, Table 5 lists the number of symmetry and Wilf classes in  $B_k$  and  $S_k$  for  $k \leq 5$ , respectively:

	$B_1$	$S_1$	$B_2$	$S_2$	$B_3$	$S_3$	$B_4$	$S_4$	$B_5$	$S_5$
# symm. classes	1	1	2	1	6	2	40	7	284	23
# Wilf classes	1	1	1	1	2	1	14	3	$\begin{smallmatrix} \geq 58 \\ \leq 137 \end{smallmatrix}$	16

TABLE 5. Number of symmetry and Wilf classes (avoidance by  $B_n$  and  $S_n$ )

Analogously, the five letter signed patters can be divided into at most 405 and at least 305 different classes when regarding their avoidance by signed involutions. The lower bound is obtained by comparing the size of  $SI_9(\tau)$ .

	$B_1$	$S_1$	$B_2$	$S_2$	$B_3$	$S_3$	$B_4$	$S_4$	$B_5$	$S_5$
# symm. classes	1	1	4	2	12	4	78	13	566	45
# Wilf classes	1	1	2	1	6	2	50	8	$\geq 305$ $\leq 405$	?

TABLE 6. Number of symmetry and Wilf classes (avoidance by  $SI_n$  and  $I_n$ )

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